Introduction to Determining Power Law Relationships

Content Discussion and Activities

PHYS 104L

1 Goal

The goal of this week's activities is to expand on a foundational understanding and comfort in modeling and testing relationships between different physical quantities. The focus will be on developing a graphical technique for determining the best power law relationship between two physical quantities based on experimental data.

2 Introduction

We have already gained experience with graphing data and using a linear regression calculation to determine the coefficients (slope and y-intercept) of the best fit line to a set of data. We have also seen how to graph and analyze measured data to test an expected relationship to see if it is consistent with the measured data or not. We could use the predicted relationship to identify a way to plot the data such that it would be expected to result in a linear graph. A linear regression calculation can be performed and the resulting best fit line coefficients can be compared with predicted values based on the expected relationship. If the best fit line coefficients (depending only on the measured data) are in agreement with the predicted values (depending only on the expected model), then the data is consistent with that model. We have looked at several situations where this process has been followed.

For today, our goal is slightly different. What if we believe two physical quantities should be related to each other (the value of one quantity depends on the value of the other) but we don't have a prediction for what that model is? We would like to be able to measure some data and use that data to determine the relationship that would best fit the data. Certainly, such a technique would come in handy when studying physical systems and trying to determine how quantities are related to each other. Such a technique could also be useful for determining, if multiple models for a system exist, which model best fits the experimental data.

Let's go over the method we will use. We'll assume two quantities, X and Y, are related to each other. Furthermore, we will assume that these quantities are related to each other by a power law relationship

$$Y = AX^n$$

Where A and n are constants. If we had a theory predicting values for A and n, we would already have a predicted/expected model. We will assume we don't have any already expected values for A and n. We will assume we have collected experimental data for corresponding values of X and Y. We could use that data to construct a graph of Y vs X and see what it looked like, however, we wouldn't have any predictions for what it would look like. If that graph turned out to be linear, we might believe that Y was linearly related to X. In other words, we would think n is equal to 1 in our power law relationship. We could perform a linear regression calculation to determine a value for A (the slope of the best fit line) and even check to see if there appeared to be a nonzero y-intercept.

On the other hand, it could be that our graph of Y vs X doesn't look linear, rather, it results in a curved graph. This result would tell us than n is not equal to 1 in a power law relationship between X and Y. We could try to guess what the value for n is. For example we could make a graph of Y vs X^2 to see if n is 2. If this graph turns out to be linear, we might decide n=2 is a good fit to the data. If not, we could try n=3. We could try n=0.5. We could try n=-4. We could keep guessing different values for n until we finally did get a graph that looked like a straight line. Once we found one that worked, performing a linear regression calculation would determine a value for A and check for any nonzero y-intercept. In this way, by trial and error, we could eventually stumble upon the values for A and n which described a power law relationship between Y and X consistent with the measured data.

However, there are some problems with this process. For one, we are having to GUESS at what the value for n might be. We might be able to make an educated guess, however, it might be difficult for us to guess just what the value for n is that we need. Guessing is never a very reliably efficient method to use. A second problem is that we would have to, based on feel or instinct, decide when our graph looked linear enough to say it was a good, straight line. For example, we might guess n=3 and get a graph that looks pretty linear. How do we know that n=4 wouldn't result in an even straighter line graph? The only way to know would be to check each and every possible value for n, then identify the one that gave the best straight line graph. Again, not reliable or efficient. What we really want is a technique allowing us to use the measured data to determine the best possible value for n for a power law relationship fitting that data. We have just such a method, however, it will involve using logarithms along with the other skills we have already covered in PHYS 104L this semester.

3A Discussion - Determining Coefficients for a Power Law Relationship

Recall, we are expecting that there is some power law relationship between the two quantities X and Y such as

$$Y = AX^n$$

We want to try and determine the best possible values for A and n. If we could determine a value for n, we already know what to do from there; we would make a graph of Y vs X^n and perform a linear regression calculation to determine the slope and y-intercept values for the resulting best fit line. All we really need to do, then, is come up with a way of determining the best value for n based on our data, without having to do a bunch of guessing or trial and error. We can do so using our graphical skills, however, we need to do a little bit of math to identify just what graph we want to make to help us out. Remember, our power law relationship is an equation. We can do some math to change the form of that equation. Let's start by taking the natural log (ln) of both sides

$$ln(Y) = ln(AX^n)$$

Now we will use a property of natural logs which you may remember from high school. It turns out that the natural log of a product is equal to the sum of their individual natural logs;

$$ln(AB) = ln(A) + ln(B)$$

Applying this to the right hand side of our prior equation gives us

$$ln(Y) = ln(A) + ln(X^n) = ln(X^n) + ln(A)$$

The order in which we add terms doesn't matter. A second property of natural logs is that

$$ln(B^C) = Cln(B)$$

Applying this to the last version we had results in

$$ln(Y) = nln(X) + ln(A)$$

This is the form we are looking for. n is some constant we are trying to determine. A is also a constant which means ln(A) is also a constant. Rather than considering a graph of Y vs X, what if we were to take the ln of all of our measured X values and also take the ln of all of our corresponding measured values for Y and construct a graph of ln(Y) vs ln(X)? If ln(Y) is our "y" for the vertical axis and ln(X) is our "x" for the horizontal axis, can you recognize the prior equation as having the form of the general equation of a straight line, "y" = m "x" + b, where the expected value for the slope of the best fit line is n, the power we are looking for in the power law relationship between Y and X. In addition, the y-intercept is expected to be ln(A), the natural log of the proportionality constant A in the power law relationship we are trying to determine. Here, then, is our method for determining the best possible value to use for n, based on the measured data. Construct a graph of ln(Y) vs ln(X), perform a linear regression calculation to get a result for the slope of the best fit line. The range for this result should contain the value we want to use for n. Now that we have a value for n, we can make a graph of Y vs X^n and complete a second linear regression calculation to determine the coefficients of the best fit line, and, hence, our best power law relationship between Y and X. Notice, we would end up having to make two graphs. First, a ln(Y) vs ln(X) graph just to determine the best value to use for n, then a second graph of Y vs X^n to come up with the coefficients for a Y vs X^n relationship.

As an FYI, you can't take the ln of something with units. As a result, technically, you have to make sure all your units cancel out on both sides of a relationship before taking the ln. This means the values you plot have no units (unitless) when you make a ln-ln graph. This is true for both the horizontal and vertical axis.

3B1 Activity – Determining Coefficients for a Power Law Relationship

The table below gives some actual student data for pendulum length and pendulum period from the pendulum activity several weeks ago.

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Length (cm)	Period (sec)
10	0.63
28	1.04
46	1.35
64	1.61
82	1.80
100	1.99

We also can recall from that lab that if the period is measured in seconds and the length is measured in centimeters, the theoretical model describing how the period is related to the length of a pendulum is

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where g is the acceleration due to gravity, 980 cm/s^2. In this case, we already have a predicted model for the relationship between T and L.

1.) Using the above data, construct a graph of ln(T) vs ln(L). Since we are expecting T and L to be related to each other via a power law relationship, this graph should appear to be quite linear. Perform a linear regression calculation to determine the slope and y-intercept of the best fit line.
2.) We can also use the known relationship to predict what we would expect the slope and y-intercept of this best fit line to be. The slope should be the power in the power law relationship; in this case, n = ½. The y-intercept should be the natural log of the proportionality constant A. In this case, that proportionality constant is $\frac{2\pi}{\sqrt{g}}$. Calculate a numerical value for the expected y-intercept and see if the slope and y-intercept of the best fit line are actually in agreement with the expected values or not. You should have a graph, regression coefficients, and two four-line summaries; one for the slope and one for the y-intercept of the best fit line. Print a copy of this graph to turn in with your activity sheet.

4A Discussion – The Saxon Bowls.

You will now study a system where we do not have a theoretically expected model in advance; the Saxon Bowls. The Saxon Bowl is a timekeeping device. You can take a bowl with a hole drilled in its bottom and place it at the top of a container of water. If you release the bowl, it will start to fill with water, eventually sinking below the water level and going to the bottom of the container. We will define the time interval of interest, T, to be the time from when the bowl is released until it just sinks under the top of the water level. You can measure this time interval using a stopwatch. As you might guess, if you had another bowl, same size, shape and weight, with a different sized hole drilled in it, this bowl would take a different amount of time, T, to sink. If the hole is bigger, it will sink quicker and if the hole is smaller, the bowl will take longer to sink. You can measure the diameter, D, of the hole in the bottom of the bowl. We will guess that there exists a power law relationship between T and D

$$T = AD^n$$

and we want to determine values for A and n based on measured data.

4B Activity – The Saxon Bowls.

1.) At your lab table, there should be a container partially filled with water. There should also be a set of Saxon Bowls at the front of the room.
2.) Grab one of the bowls to use for data collection. Each bowl is numbered so record the bowl number on your activity sheet.
3.) Use a Vernier caliper to measure the diameter of the hole in the bowl in units of centimeters. Record this value on your activity sheet.
4.) Dip the bowl in the water so that it is wet. Empty the bowl and now place it right at the top of the water surface. Using a stopwatch, measure the time the bowl takes to sink just below the water level after it is released. Record this time. Repeat this measurement two more times for a total of three measurements. Record all three values on the activity sheet.
5.) Calculate the average of these three values and record it as well on the activity sheet.
6.) Repeat steps 2-5 for five additional bowls. Make sure the bowl numbers are spread out and span the full range of hole sizes. Don't have them all be close together.
7.) Make a graph of ln(T) vs ln(D). Perform a linear regression calculation and list the result and range for both the slope and y-intercept of the best fit line in two-line summaries on the graph below the regression coefficients. Use the range for the slope to identify your best guess for the correct value to use for n. Hint: you would not expect the value for n to be something like 6.418320847! Rather, you would expect n to likely be either an integer (-3, -2, -1, 1, 2, 3, ….), a half integer (… -3/2, -1/2, ½, 3/2, 5/2, ….), or possibly some fractional integer(7/5, 3/8, 2/3, 7/3…). n could be any value within your result range so try to identify the most likely value, not necessarily the best estimate from your graph.

8.) Now create a graph of T vs D^n. Perform a linear regression calculation to determine the slope and y-intercept of the best fit line. List a result and range for each of these, properly rounded with units, in two-line summary format.

9.) At the very bottom of your graph page, write out the power law relationship, based on your data, between T and D. This should be an equation with a T and a D and then determined constants from your last graph.

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Activity Data Sheet

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Activity 3B

Show how you calculated the expected value for the y-intercept of your graph.

Activity 4B

Bowl	Diameter	Time 1	Time 2	Time 3	Avg Time
#	(cm)	(Sec)	(Sec)	(Sec)	(Sec)

Write out your determined relationship between T and D here too; include uncertainties and units.