Introduction to Calculations and Uncertainties

Content Discussion and Activities

1 Goal

The goal of this week's activities is to provide a foundational understanding regarding calculations using measured quantities and the resulting uncertainty in calculated values. Units in calculations, the uncertainty in calculated values and use of significant figures to estimate uncertainties when one is limited only by reading errors are primary topics to be covered.

2 Introduction

We have discussed our need, in science, to determine the values for physical quantities. Sometimes, we are able to directly measure the value of a physical quantity of interest as in last week's activities. Other times, we are not able to directly measure something, however, we can measure other quantities and use those measurements to calculate a value for the quantity of interest. The ability to perform and understand calculations with measured values is crucial for a scientist. Developing a better understanding of these skills is our goal this week.

3A1 Discussion - Calculations and Units when Multiplying or Dividing

Usually, when people think of performing calculations, they think of plugging numbers into a calculator. Doing computations with numbers is certainly part of doing calculations. However, as we found last week, a number by itself isn't enough to describe a physical quantity; we needed to include the appropriate units along with that number. The number and the units have to both be included to describe a measured quantity. It turns out, the units have to follow all the same calculations that the numbers do. We have already seen one example of this when doing unit conversions last week. Recall, we could convert a distance of 219 cm into units of inches as follows:

$$219 \text{ cm} * \frac{1.00 inch}{2.54 cm} = 86.2 \text{ in}$$

In order to get the final value, one would have to do calculations with the numbers involved; enter 219 into a calculator, multiply by 1.00, and then divide by 2.54 to get 86.2. We may have glossed over the fact that the units had to complete all the same operations. We also took the units of cm from the distance, multiplied by inches and then divided by cm. The cm units canceled out leaving only units of inches. Hence our final result for the distance of 86.2 inches, which is both a number and a set of units. Anytime you multiply or divide 2 physical quantities you must do the math with the numbers as well as the units. You don't plug the units into your calculator, rather, you look to see if there is any cancellation and whatever doesn't cancel is left as the units that go with the numerical value you calculated.

As an example, consider the quantity speed. One can calculate the average speed of an object by dividing the distance travelled by the time it took to travel that distance; $speed = \frac{distance}{time\,interval}$. Suppose you know an object traveled a distance of 5.4 meters in a time interval of 1.2 seconds and you want to calculate the average speed of the object during that time interval.

$$\text{Speed} = \frac{distance}{time\,interval} = \frac{5.4m}{1.2sec} = 4.5\frac{m}{sec}$$

Notice how the numbers were used to calculate a final numerical value, but the units also completed those math operations too. We are still left with units of meters in the numerator and seconds in the denominator.

3B1 Activity – Calculations and Units when Multiplying or Dividing

1.) On your activity sheet, several different calculations are listed. Determine the units for the final result based on the given information for each situation.

3A2 Discussion - Calculations and Units when Adding or Subtracting

Just as when multiplying or dividing, the units also have to complete the same math operations as the numbers do when adding or subtracting. However, there is an additional constraint on these calculations. While you can multiply or divide two quantities with different units, you are only able to add or subtract quantities if they have the same units. In other words, you can only add meters to meters and seconds to seconds. It does not make physical sense to add 4 meters to 3 seconds. Even though you can add the numbers 4 and 3 together to get 7, what would the units be for that answer? When you add or subtract quantities, the final answer must have a single set of units and those units must match the units of both of the original quantities. This means the original quantities must have the same units. You cannot add a length and a time interval or an area to a mass. If you have two lengths, but they are described with different units, you cannot add those values, either. You first have to convert one or both of those measured values into the same set of units, then you can add or subtract them such that the result can have a physical meaning. Consider adding two time intervals, one is 45 minutes and the second is 2.5 hours. We want to find the sum of those two time intervals. Does it make sense that the answer is NOT 47.5? We first have to convert either the first time interval into hours or the second time interval into seconds. Then we would be able to add them. 2.50 hours = 150 minutes so adding the two time intervals together would give 45 min + 150 min = 195 minutes. Alternatively, 45 min is equal to 0.75 hours so we could do 0.75 hours + 2.50 hours = 3.25 hours. Try taking the first result (195 minutes) and doing a unit conversion to convert it to hours. What should you get?

3B2 Activity – Calculations and Units when adding or Subtracting

1.) On your activity sheet, several calculations involving addition and/or subtraction are described. Perform these calculations and list the result being sure to include the correct units on your answer. Show your work on the activity sheet.

4A Discussion – Uncertainty in Calculated Values

Since we know all measured values have uncertainties associated with them, it should make sense that anything we calculate using measured values is going to have uncertainty in it as well. Bigger uncertainties in our measured quantities will lead to larger uncertainties in calculated quantities while smaller measurement uncertainties will lead to smaller uncertainties in calculated quantities. We will cover some ways of trying to determine the uncertainty in a calculated quantity. For now, we will assume the reading error for a measurement is our uncertainty in that measured value. Remember that measurement actually corresponds to a range of values described as Measured Value $\pm$ Reading Error. Our best estimate for the value of what we are trying to calculate would be what we calculate using the measured values for our measured quantities. We know the actual value for those quantities, however, could be anywhere within the measurement range. We can use the upper and lower limits of the "measurement ranges" to calculate the worst case scenario upper and lower limits for our calculated value. As an example, let's assume we want to calculate the area of a rectangular paper. We measured the length of the paper to be 23 cm and the width to be 14 cm both with reading errors of 1 cm. Area = length*width, so our best estimate for the area of the paper is

$$\text{Area} = \text{length} * \text{width} = 23 \text{ cm} * 14 \text{ cm} = 322 \text{ cm}^2$$

Notice again, the numbers were multiplied and the units were multiplied as well. This would be our best estimate for the value of the area of the paper. We believe the actual length is between 24 and 22 cm and the actual width is between 15 and 13 cm. If we wanted to calculate the "worst case scenario" biggest value for the area, we would use the upper limits of both of these ranges for length and width

$$\text{Area}_{\text{upper limit}} = 24 \text{ cm} * 15 \text{ cm} = 360 \text{ cm}^2 \quad \text{and} \quad \text{Area}_{\text{lower limit}} = 22 \text{ cm} * 13 \text{ cm} = 286 \text{ cm}^2$$

would be the lower limit for the area of the paper. We expect the area should be between 360 and 286 cm^2 with our best estimate as 322 cm^2. This range is reasonably well described as

$$\text{Area} = 320 \pm 40 \text{ cm}^2$$

We've rounded the answer this way since the upper and lower limits are about 40 units different from the best estimate, we know its uncertainty is in the tens place, so it makes sense to round the best estimate to the nearest 10 place, and likewise for the uncertainty. We can be pretty confident that the actual area of that paper is within 40cm^2 of 320 cm^2. Remember, using the upper limits we were calculating the "worst case scenario" so we are probably closer than 40 cm^2, but that's the furthest off we would expect to be.

As a second example, revisit our calculated speed from before. With a distance of 5.4 m and a time interval of 1.2 seconds, we calculated a speed of $4.5 \frac{m}{s}$. We'll assume reading errors of 0.1 m and 0.1 seconds. This time, to get our upper limit for our speed range, we need to use the upper limit for the distance and the lower limit for the time interval (smaller denominator would give a bigger result).

$$\text{Speed}_{\text{upper limit}} = \frac{5.5m}{1.1s} = 5.0 \frac{m}{s} \quad \text{and similarly the lower limit is} \quad \text{Speed}_{\text{lower limit}} = \frac{5.3m}{1.3s} = 4.1 \frac{m}{s}$$

This range could best be described as Speed $= 4.5 \pm 0.5 \frac{m}{s}$, which would give us a worst case scenario estimate of the range in which we think the actual speed of the object is within. Again, the best estimate is probably closer than $0.5 \frac{m}{s}$ to the actual value, but that would be the worst case scenario.

4B Activity – Uncertainty in Calculated Values

1.) You should have a block, a ruler, and a caliper. Use the ruler to measure the length, width and height of the block in cm. Record these values on your activity sheet. Also record the reading error for these measurements.

2.) Remeasure these three quantities using the caliper. Record these values on your activity sheet along with the reading error for them.

3.) Calculate the Best Estimate of the Total Outside Dimensions (TOD = Length + Width + Height) for the block using the measurements with the ruler. List this value on your activity sheet and show your work.

4.) Calculate the "worst case scenario" values for the upper and lower limits of the calculated TOD. List these values on your activity sheet and show your work.

5.) Give the TOD of the block in the form of Best Estimate $\pm$ Uncertainty based on the calculated values from steps 3 and 4 as done in the examples above. List this result on the activity sheet and explain how you determined the uncertainty to use.

6.) Repeat steps 3-5 using the measured values from the caliper.

7.) Repeat steps 3-5 using the height from your ruler data and the length and width values from your caliper data.

8.) Knowing that Volume = Length*Width*Height for a rectangular block, calculate your Best Estimate for the volume of the block using your measured values using the ruler. List this value on your activity sheet and show your work.

9.) Calculate the "worst case scenario" values for the upper and lower limits of the calculated volume. List these values on your activity sheet and show your work.

10.) Give the volume of the block in the form of Best Estimate $\pm$ Uncertainty based on the calculated values from steps 8 and 9. List this result on the activity sheet and explain how you determined the uncertainty to use.

11.) Repeat steps 8-10 using the measured values from the caliper.

12.) Repeat steps 8-10 using the height from your ruler data and the length and width values from your caliper data.

13.) How did the uncertainties for these three calculated Volume results compare? What can you learn from this?

5A Discussion – Significant Figures to Estimate Uncertainty

The method of using upper and lower limits of measurements to calculate "worst case scenario" upper and lower limits for calculated values is one way of calculating uncertainties. As you may have noticed, however, it can take some time. Significant Figures is an alternative technique you can use to estimate the uncertainty in a calculated values. What does a scientist mean when they talk about "Significant Figures?" Back in our example where we calculated an area, we took 23 cm * 14 cm and got 322 cm^2. We assumed there was a reading error of $\pm$ 1 cm in each of those measured values. Both of those measured values are said to have **2 significant figures**; 2 figures (or digits) were significant in those measurements. If we had the same reading error and measured a distance of 1428 cm, that measurement would have 4 significant figures. If we had a reading error of $\pm$ 100 cm and a measured value of 27,400 cm, that measurement would have 3 significant figures. The two zeros are just placeholders, not significant figures in that measurement. If you know what digit has uncertainty in it, that digit plus any digits to the LEFT of it are significant figures, and any digits to the right are not significant figures. In this way, one can assess the number of significant figures any measured value contains. Going back to our calculated area, it turned out that, based on our upper and lower limits, there was an uncertainty of 40 cm^2 in that calculated value; there was uncertainty in the "10s" digit place. From what we just covered, the "10s" digit was a significant figure plus any digit to the left of it (the 3 in the 100s place) is also significant. The 2 in the "1s" digit place is not significant. This is where the label "significant figures" comes from. If there is uncertainty in the "10s" digit place, then the 2 in the "1s" place has no significance because the uncertainty in the calculated value is much bigger than that digit's place.

Once we can determine the number of significant figures in measured values, we can use the **Rules for Significant Figures in Calculations** to quickly estimate the number of significant figures in a calculated value. There is one rule for multiplication and division and a second rule for addition and subtraction.

Multiplication and Division – the number of significant figures in the calculated values is the same as the **smallest number** of significant figures in any individual value used in the calculation.

Addition and Subtraction – the last significant figure in the calculated value is determined by the measured value with the **highest** last significant figure.

Let's see if these make sense. In calculating the area, we multiplied two values each of which had 2 significant figures. That means our final answer should only have had 2 significant figures. In other words, we expect there to be uncertainty in the second digit place of our answer. This is just what we found as it turned out there was uncertainty in the "10s" digit place giving us two significant figures in the answer. Using the rules for significant figures, we could have quickly estimated that the 2 in the "1s" digit place wasn't significant and we could just round off to the nearest "10s" digit place. We don't know how much uncertainty there was, it could be ± 10 cm^2, ± 30 cm^2, or some other value, but we know the uncertainty should be in that "10s" digit place. If we estimated ± 50 cm^2, that would cover the full range of possible "10s" digits (corresponding to a "worst case scenario" situation) so we could have quickly estimated a calculated area of 320 ± 50 cm^2. This technique is much quicker than the upper and lower limit method and will give us results that match up pretty well as can be seen in this example. Often times we may not explicitly write down the uncertainty, rather, just use significant figures to be sure we know what digit place to round the final answer to so that we can correctly indicate which digit has uncertainty in it.

Just as an example for addition, what if we wanted to calculate the Total Outer Dimensions of that paper from the example. TOD = Length + Width (assuming negligible height) so we would get 23 cm + 14 cm = 37 cm. Both of those measurements had their least significant figure in the "1s" digit place. This means the answer would have uncertainty in the "1s" digit place. There is uncertainty in the 7, but it is a significant figure. What if our measured values had been 23 cm and 14.26 cm. When calculating the TOD, we would get 37.26 on our calculator. The length measurement had its least significant figure in the "1s" place and the width had its least significant figure in the ".01s" digit place. The significant figures in the answer are determined by the largest least significant figure in any of the values that were added so our answer's least significant figure would be in the "1s" place. We would round it off to TOD = 37 cm. The 2 and the 6 digits in the ".1s" and ".01s" digits places aren't significant because of the ± 1 cm uncertainty in the length. The rules for significant figures give us a handy way of identifying how to round off calculated values.

5B Activity – Significant Figures to Estimate Uncertainty

1.) Revisit the six calculations from the prior activity. Use the rules for significant figures to determine which digits in your calculated values are significant and correctly round them. For each of the calculations, list the unrounded Best Estimate, the Best Estimate rounded off using the rules for significant figures, and explain how you used the significant figures rules to round the answers.

Introduction to Calculations and Uncertainties

Activity Data Sheet

Activity 3B1

1.) Area = Length * Width　　　　　　　　Area Units = _____

Length and Width measured in meters (m)

2.) Volume = $4/3\pi r^3$　　　　　　　　　　Volume Units = _____

Radius measured in centimeters (cm)

3.) Speed = distance/time interval　　　　Speed Units = _____

Distance in kilometers (km) and time in minutes (min)

4.) Current = Charge/Time　　　　　　　Current Units = _____

Charge in Coulombs (C) and time in seconds (s)

5.) Crop Yield = Crop Harvested / Land　　Crop Yield Units = _____

Crop Harvested in bushels (bu) and land in acres (acre)

Activity 3B2

1.) Calculate the total sum of the following time intervals; 18 days, 3.0 weeks, 2.0 months

2.) Calculate the total sum of the following three distances; 6.26 meters, 382 cm, 114 inches

3.) Calculate the total sum of these volumes; 0.573 liters, 286 cubic cm (cm^3), 0.0975 gallons

Activity 4B

	Length (cm)	Width (cm)	Height (cm)	Reading Error (cm)
Ruler				
Caliper				

3.) Calculate TOD using ruler data:

4.) Calculate TOD upper and lower limits

5.) TOD_{ruler} = _____ ± _____

6.) Calculate TOD using caliper data:

6.) Calculate TOD upper and lower limits

6.) $TOD_{caliper}$ = _____ ± _____

7.) Calculate TOD using mixed data:

7.) Calculate TOD upper and lower limits

7.) TOD$_{mixed}$ = _____ ± _____

8.) Calculate Volume using ruler data:

9.) Calculate volume upper and lower limits

10.) Volume$_{ruler}$ = _____ ± _____

11.) Calculate Volume using caliper data:

11.) Calculate volume upper and lower limits

11.) Volume$_{caliper}$ = _____ ± _____

12.) Calculate Volume using mixed data:

12.) Calculate volume upper and lower limits

12.) Volume$_{mixed}$ = _____ ± _____

13.) How do the uncertainties compare?

Activity 5B

Using Ruler Data:

TOD unrounded = _____   Explain:

TOD rounded = _____

Volume unrounded = _____   Explain

Volume rounded = _____

Using Caliper Data:

TOD unrounded = _____   Explain:

TOD rounded = _____

Volume unrounded = _____   Explain

Volume rounded = _____

Using Mixed Data:

TOD unrounded = _____   Explain:

TOD rounded = _____

Volume unrounded = _____   Explain

Volume rounded = _____

Are the rounded values using the rules for significant figures consistent with the results from Activity 4B? Explain.