Simple Harmonic Motion

1   Object

To determine the period of motion of objects that are executing simple harmonic motion and to check the theoretical prediction of such periods.

2   Apparatus

Assorted weights and spheres, clamps, meter stick, spring, stand, stopwatch, protractor, string, motion detector and interfaced computer.

3   Theory

Simple harmonic motion refers to a type of movement that is found in a number of apparently dissimilar situations. Three such situations will be studied in this experiment. Harmonic motion is that which repeats itself at certain intervals; one such interval is called the period of motion. A specific type of periodic motion, simple harmonic motion, has a rather straightforward mathematical representation,

$$x = A \cos (\omega t + \delta) \tag{1}$$

where A is the amplitude of motion, ω is the angular frequency, t is the elapsed time, and δ is a phase factor identifying at what point in the cycle we chose $t = 0$. The angular frequency is related to the period through the relationship

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \tag{2}$$

If an object undergoes motion as described above, the object's velocity and acceleration as functions of time will be,

$$v = -A\omega \sin (\omega t + \delta)$$
$$a = -A\omega^2 \cos (\omega t + \delta) \tag{3}$$

One can now see the necessary condition for an object to undergo simple harmonic motion. Notice in equation 3, that the acceleration as a function of time is proportional to $A \cos (\omega t + \delta)$ which is nothing other than the position of the object as a function of time. Substitution yields,

$$a = -\omega^2 x \tag{4}$$

An object undergoing simple harmonic motion must always have an acceleration directed in the opposite direction of the object's displacement from equilibrium and the magnitude of the acceleration is always proportional to the objects displacement. In other words, the acceleration is always directed towards equilibrium and gets proportionately larger in magnitude as the object moves further from equilibrium. Since the net force is always in the direction of an object's acceleration, it is often said that simple harmonic motion requires a restoring net force, a net force directed towards equilibrium, which is proportional to the displacement from equilibrium. If one can show

that the acceleration is equal to a negative constant times the displacement from equilibrium, the motion will be simple harmonic and the angular frequency of the motion is the square root of the magnitude of that constant.

3.1 Part 1 Mass on a Spring

It was found experimentally by Robert Hooke that to within elastic limits (no permanent deformation) a spring will be stretched a distance proportional to the applied force. Mathematically this is stated as

$$F = -k \cdot stretch \tag{5}$$

where F is the force applied to the spring and k is a constant of proportionality, often called the *spring constant*, which is a measure of the elasticity of the spring. The negative sign shows that the force exerted by the spring is always in the opposite direction of the stretch of the spring. If a mass is at rest, suspended from the spring, the mass is in static equilibrium. The only forces acting on the mass are a gravitational force down, mg, and the spring force up, kx_o, where x_0 is the
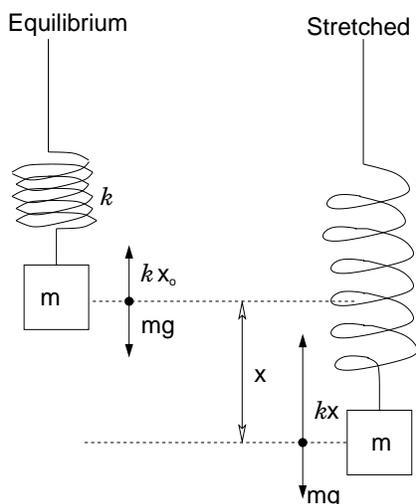
stretch of the spring at equilibrium. If the mass is now displaced from that equilibrium position by some displacement x, the net force acting on the mass would be the extra force due to this further displacement x. Newton's second law would yield

$$F_{net} = -kx = ma \quad or \quad a = -\frac{k}{m}x \tag{6}$$

after solving for the acceleration. Notice that this is just the form which indicates motion which is simple harmonic. The resulting displacement of the mass from the equilibrium position as a function of time must be given by equation 1. The constant in equation 6 can be used in conjunction with equation 2 to show that for the oscillating motion of a mass on a vertical spring

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{7}$$

3.2 Part 2 Simple Pendulum

A similar type of reasoning will allow one to derive the period of motion for a simple pendulum. An example of a simple pendulum is a mass attached to a massless string of length, ℓ, which in turn is attached to a fixed object. Assume the string to be pulled away from the vertical by a small angle θ; the mass attached to the string is displaced through a circular arc of length s. Applying
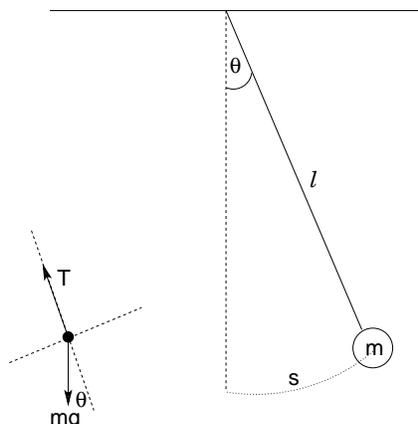
Newton's 2nd law to the object in the direction of motion gives

$$F_{net} = ma = mg\sin(\theta) \tag{8}$$

for the magnitude of the net force acting on the mass in the direction of motion. Notice that the net force is always directed toward the equilibrium position at the bottom of the pendulum's swing. Also, for small angles, $\sin(\theta) \simeq \theta = \frac{s}{\ell}$, where s is the displacement of the mass from equilibrium. Solving equation 8 for the acceleration a yields

$$a = -\frac{gs}{\ell} \tag{9}$$

where the minus sign indicates that the net force is in the opposite direction of the displacement. This result is the same form as equation 4, which was the condition for simple harmonic motion. The expected period of this motion would be
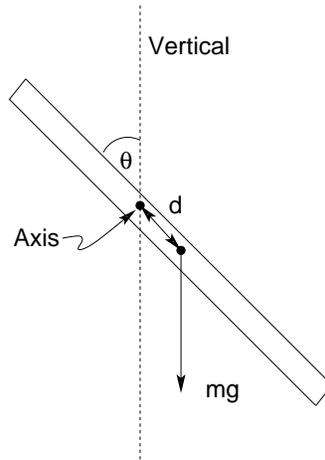
$$T = 2\pi\sqrt{\frac{\ell}{g}} \tag{10}$$

where ℓ is the length of the pendulum and g is the acceleration due to gravity.

If the angle of displacement is not small, the $\sin\theta$ cannot be reasonably approximated by θ and one has a much more complicated expression

$$T = 2\pi\sqrt{\frac{\ell}{g}}\left[1 + \frac{\left(\sin\frac{\theta}{2}\right)^2}{2^2} + \frac{1}{2^2}\frac{3^2}{4^2}\left(\sin\frac{\theta}{2}\right)^4 + \dots\right] \tag{11}$$

The ... means "and so on forever."

3.3   Part 3 Physical Pendulum

A similar result is obtained for the physical pendulum. This is a situation where an object of finite mass and size is rotating about a fixed point on the object itself. In this situation, one considers the rotational equivalent of Newton's Second Law

$$\tau_{net} = I\alpha \tag{12}$$

where τ_{net} is the net torque acting on the object, I is the object's moment of inertia about the fixed point and α is the angular acceleration. Once again, it is clear that the torque will be a restoring torque since any displacement will produce motion back toward a vertical orientation of d (see figure below). Note that

$$\tau_{net} = Fr_\perp = mgr_\perp = mgd\sin(\theta) \tag{13}$$

where m is the mass of the object, P is the pivot point, C is the center of mass, d is the distance between those two points and θ is the angle of displacement of d from the vertical. If once again, the angle of displacement is small, $\sin\theta \simeq \theta$ and

$$\alpha = -\frac{mgd}{I}\theta \tag{14}$$

where again, the negative sign is due to the direction of the net torque. The restoring torque is proportional to the angular displacement and one will have simple harmonic motion. Again, by comparing equation 14 with equation 4 and using equation 2, the period of this simple harmonic motion is determined as

$$T = 2\pi\sqrt{\frac{I}{mgd}} \tag{15}$$

where the only term to be determined is I, the moment of inertia of the physical pendulum about the pivot point P. The moment of inertia is the rotational equivalent for the mass of an object. It will, in general, depend on the mass of the object and the location of the pivot point. About the object's center of mass, the moment of inertia will be a minimum; however, as the pivot point recedes from the center of mass, the object will better resist changes in its rotational motion meaning I has increased. The parallel axis theorem allows us to find the moment of inertia of an object about some point that isn't its center of mass by

$$I_P = I_{c.m.} + md^2 \tag{16}$$

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where m is the object's mass, d is the distance between point P and the center of mass and the moment of inertia about the center of mass is known. For the bar used in part 3 of the experiment, the moment of inertia is then

$$I_P = \frac{mL^2}{12} + md^2 \tag{17}$$

where L is the length of the bar. Again, if the angle of initial displacement is not small, a correction similar to that made for the case of a simple pendulum is in order,

$$T = 2\pi\sqrt{\frac{I}{mgd}}\left[1 + \frac{\left(\sin\frac{\theta}{2}\right)^2}{2^2} + \frac{1}{2^2}\frac{3^2}{4^2}\left(\sin\frac{\theta}{2}\right)^4 + \dots\right] \tag{18}$$

4   Procedure

1. Set up a simple pendulum. Pull the string aside so that it makes an angle of $30°$ with the vertical. This is called the initial angle. Let the object swing and record the time for ten full oscillations. Record the length of the string. This is the distance from the point where the string is attached to the center of mass of the object used as the pendulum. Repeat this four times for the same object, same length, and same initial angle.

   Change the initial angle to $5°$ and repeat the procedure. Then repeat the above procedure for initial angles of $10°$, $15°$, and $45°$.

2. To determine the spring constant k of a spring, hang a $150\,g$ mass on a spring, then record the additional stretching due to $30\,g$ more, then $60\,g$, $90\,g$, $120\,g$, $150\,g$, and $180\,g$.

3. Setup the computer to take data using the motion detector. You may load settings from the file *shm*. Remember, the yellow motion detector plug goes in the leftmost input. Place the motion detector on the bench top, facing up, and orient the spring so that it is hanging directly above the motion detector. Place $300\,g$ total on the end of the spring. Tape the masses to the mass hanger, and also tape the mass hanger hook to the bottom of the spring, so that nothing can fall off and possibly damage the motion sensor. Adjust the support bar so that the equilibrium position is at least $50\,cm$ above the motion detector. Record graphs for several periods of the motion resulting from displacing the mass $10\,cm$ and releasing it. Make copies of these graphs. Take a good look at these three graphs to better understand the relationships between the physical quantities position, velocity and acceleration for an object undergoing simple harmonic motion. Do the graphs look sinusoidal?

4. Record the following information from the graphs. When the mass is at its maximum positive displacement, what is its velocity, what is its acceleration? When the mass passes back through equilibrium, what is its velocity and acceleration. Repeat at the maximum negative displacement and at the following equilibrium point. From the graphs, determine the period of the simple harmonic motion. Measure and record the period at 3 different points for each of the three graphs. Also measure and record the amplitude of the motion, the maximum speed of the object and the maximum acceleration.

5. Using a stopwatch, record the time for 20 oscillations of the mass on the spring.

   Repeat this timing process for five different masses, recording the time for twenty oscillations for each mass.

6. Construct a physical pendulum so that it is able to swing about a pivot point. The physical pendulum in this case is a meter stick. Record the length and mass of this object. Also record the distance from its pivot point to its center. Then pull it from the vertical so that it makes an initial angle of 30°. Release it and time 10 oscillations. Repeat the above measurement for an initial angle of 5°.

Change the pivot point of the object and repeat the above procedure. Record the location of the new pivot point. Then repeat the entire process for three other pivot points.

5   Calculations

1. Calculate the theoretical period for one oscillation of the simple pendulum by using equation 10. Compare this with the experimentally (stopwatch) determined period (mean, error and range) as measured directly.

2. Use your data from procedure step 2 to determine the spring constant of your spring. To do this, construct a graph of added weight (weight of the additional mass added to the initial $150\,g$) vs additional stretch. Does this graph appear to be linear? Should it? Determine the best fit line to the data using a linear regression program. According to equation 5, the slope of this line should be related to spring constant of the spring. Determine k from your best fit line. Is the y intercept of the best fit line what it should be? Is Hooke's law valid for your spring?

3. Calculate the average of the 9 values for the period of the oscillating mass measured from your motion detector graphs in step 4 of the procedure. Using this value for the period and your measured amplitude, calculate the expected maximum speed and the expected maximum acceleration as predicted by equation 3. How do these values compare to those measured from the graphs?

4. Using the values for T from the stopwatch measurements, construct a graph of period vs mass. Also construct a graph of T^2 vs total mass on spring. Do these graphs look like they should? According to equation 7 the period squared graph should be a straight line with slope of $\frac{4\pi^2}{k}$. Using the value of k extracted from the graph in step 2 of calculations, calculate a theoretical value for the slope of your new graph. Use a linear regression program to find the best fit line to the data. Do the slope and y intercept of this line agree with your expected values?

5. Turning to the physical pendulum data, use equation 15 to calculate theoretical values for the period of your physical pendulum for each value of d. Which set of data, 5° or 30°, agrees best with the theoretical predictions? Explain.

6   Questions

1. If you chose the center of the meter stick as the pivot point, what would the period be? Give a physical (non-mathematical) explanation.

2. If spring A has a spring constant which is 8 times spring B's spring constant, what is the ratio of their periods? Which is the stiffer spring?

3. Why doesn't your period squared vs mass graph for the spring go through the origin? Explain.

Simple Harmonic Motion Data Sheet

Part 1 - Simple Pendulum

Length (*cm*): _____

Trial	5°	10°	15°	30°	45°
1					
2					
3					
4					
5					

Which of the five initial angles should give results that most closely agree with the theoretical predictions? Why?

Part 2 - Mass on a spring

Additional Mass (*g*)	Additional Stretch (*cm*)

Added Mass (*g*)	Time 20 Osc. (*sec*)

Simple Harmonic Motion Data Sheet

Period (*sec*)				

Displacement	Velocity	Acceleration
Max. Pos. Disp		
Equil. Moving Down		
Max. Neg. Disp		
Equil. moving Up		

Amplitude	
Max. Speed	
Max Acceleration	

Part 3 - Physical Pendulum

Pendulum Mass	
Pendulum Length	

Distance From Center to pivot	Time for 10 oscillations	
	30°	5°