Error Analysis

1 Object

To become familiar with some principles of error analysis for use in later laboratories.

2 Apparatus

A plastic tub, water, Saxon Bowls, and a stopwatch.

3 Theory

In science one often makes measurements on some physical system – length, mass, a time interval, *etc.* – in order to quantify and hopefully better understand the system of interest. Errors are inherent in *any* measurement, and the experimenter needs to not only be aware of this fact, but needs some tools at the ready to deal with the errors.

3.1 Reading Errors

Each time a measurement is made using some type of instrument (meter stick, caliper, stopwatch, *etc.*) there will be an error associated with the reading of the instrument. We take the reading error of an instrument as ± one of the smallest divisions readable on the instrument. For example, if one measures a length with a meter stick, then the reading error is $\pm 1\,mm$. For a stop watch with times down to one hundredth of a second, the reading error is $\pm 0.01\,sec$.

3.2 Random Errors

Often multiple measurements of a single quantity which one would expect to remain constant will vary by an amount larger than the reading error. This variation in the measurement is the result of random error effects which sometimes cause the measurement to be too high and sometimes too low. As we will soon find, if we make enough measurements of the same quantity which is subject to small random errors, the effects of these random errors may be averaged out, leaving the average value relatively unaffected by the random error effects. The main point here is that variations in the measured values of a quantity which should remain constant are the results of *Random Errors*. What does it mean if multiple measurements of a given quantity do not scatter, but rather are indeed constant? Does this mean the measurement has no uncertainty associated with it? No. This means that the random error effects are smaller than the reading error. The uncertainty in this measured quantity is determined by the reading error, not random error effects. The uncertainty associated with a measured quantity in lab will not be less than the reading error. Typically, in this lab, the random errors associated with a measurement will be larger than the reading error and thus will dominate the uncertainty in our measurements – but not always. The student must be aware that reading errors exist and also be able to correctly account for them if necessary.

3.3 Systematic Errors

Random errors are errors for which the effects will average out after taking multiple measurements. Systematic errors are errors for which this is not the case. A *systematic error* is one which will always cause the same mis-measurement. A systematic error does not result in scatter in the data but rather a consistent over estimation or under estimation of the true value. Simply repeating a measurement many times will not uncover a systematic error affecting the results. Assuming the above is correct, any lack of agreement between theory and experimental results is due to such systematic errors. For example, maybe you are measuring the length of several salamanders, and you are using a meter stick so you can measure lengths with a reading error of $1\,mm$. Since the salamanders are different, we will expect the lengths to vary by more than this reading error because of random effects. The average of all these measurements would be the average length of this set of salamanders, and our uncertainty in this average value would be determined by random effects rather than the reading error from using the meter stick. But what if we were using a meter stick that had accidentally had the first $4.0\,cm$ cut off and we failed to notice? This would be a systematic error affecting our measurements. Every one of our measurements would be $4.0\,cm$ too small, and the average of all of these measurements would be $4.0\,cm$ too small as well. The only way we would be able to notice this systematic error would be by comparing the average of these measurements with the average of measurements made with a "good" meter stick. They should not agree. If they did agree, however, that would indicate that the random fluctuation of salamander length was larger than the $4.0\,cm$ systematic error of the shortened meter stick. So, systematic errors are permissible as long as they are smaller than the random errors affecting our measurements. It is only when the random errors are smaller than the systematic error effects that these systematic effects become noticeable.

3.4 Statistical Analysis

Throughout the semester we will be making measurements, and to attempt to deal with the errors we will often make repeated measurements of some aspect of a system. We will then take the *mean* (average) of these measurements in the hope that that is a better overall value for the particular quantity than any single measurement. The mean $\bar{x}$ of a bunch of values x_i is given by:

$$\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N} = \frac{x_1 + x_2 + x_3 + \cdots x_N}{N} \tag{1}$$

Where the symbol $\sum_{i=1}^{N}$ means "sum up the terms x_i from $i = 1$ to $i = N$." This is just math-speak for something that we all should already know how to do in our sleep; however, it helps to get used to the fancy math-speak. The mean value of x, whatever x happens to be, then is our best guess for what the most likely value of a measurement of x would give.

We aren't done yet, no-no-no! We can do more. In fact, we can not only get the mean by doing the above, but we can estimate an error associated with our mean. We do this because it can give us some measure of confidence in the mean we will quote – just how good our mean might be, and consequently how dependable our measurements were. To arrive at the error associated with the mean, we first calculate σ, the *standard deviation.* σ is calculated the following way:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(\bar{x} - x_i)^2}{N-1}} = \sqrt{\frac{(\bar{x} - x_1)^2 + (\bar{x} - x_2)^2 + \cdots + (\bar{x} - x_N)^2}{N-1}} \tag{2}$$

From this we can calculate the *standard error on the mean*, s_x, as:

$$s_x = \sigma/\sqrt{N} \tag{3}$$

and from this we can finally get the *error associated with the mean*, e_x, as:

$$e_x = 1.96\, s_x \tag{4}$$

This then is the error that we associate with the mean or average value for a measurement. When we "quote" the mean value of a measurement in a lab report, we write it as:

$$value = \bar{x} \pm e_x \tag{5}$$

3.4.1 Example

As a quick example, suppose you took data in a lab which looked like this:

Trial	$x\,(cm)$
1	14.1
2	14.7
3	15.1
4	15.4
5	14.3

We calculate the mean of x as $\bar{x} = 14.72\,cm$. If we now calculate the standard deviation, we get:

$$\begin{aligned}
\sigma_x &= \sqrt{\frac{\sum_{i=1}^{N=5}(\bar{x} - x_i)^2}{N-1}} \\
&= \sqrt{\frac{(14.72-14.1)^2 + (14.72-14.7)^2 + (14.72-15.1)^2 + (14.72-15.4)^2 + (14.72-14.3)^2}{5-1}}\,cm \\
&= 0.54037\,cm
\end{aligned}$$

Next we calculate the standard error on the mean:

$$s_x = \sigma_x/\sqrt{N} = 0.24166\,cm$$

Then the error associated with this mean is:

$$e_x = 1.96\, s_x = 0.47365\,cm \simeq 0.5\,cm$$

Note that up until the last step we kept many digits (we keep all of them that our calculator/spreadsheet gives us to prevent numerical round off errors), but in the last step above we rounded to $0.5\,cm$. This is how we want you to do things – keep all digits in intermediate steps, and round off in the last step with the answer that is important. Usually this answer is the one we want to compare to some theoretical prediction, and the intermediate steps are just a way to get to it.

The rule for rounding is this: look at the number that is the *error*, and use the first non-zero digit *only*, regardless of where that digit is with respect to the decimal point. For example, if we were to calculate an error of 0.00246, we would use 0.002 as our error. If the error was 0.362, we would use 0.4 as our error, because normal rounding applies. We then quote our actual numerical answer to the number of decimal places to match our error. So, for our example above, we quote the value as $\bar{x} = 14.7 \pm 0.5\,cm$, where the answer has one decimal place because the error is in the first place to the right of the decimal.

There is one special case where we take more than one digit for our error, and that is if the first digit is a one (1). In that case, we also take the next digit, properly rounded, and quote our error with two digits. So, if the error came out to be 0.01342, we would use 0.013 as our error, and the result should have three decimal places to match this error.

3.4.2 What does a Standard Deviation Mean?

A standard deviation gives one a measure of how widely scattered the set of data is. That is, one has made repeated measurements of a particular aspect of a system which, theoretically, should always give exactly the same result. In practice, however, this is not true, and the read values of this aspect being measured vary from one trial to the next. If one is careful, then the scatter from one measurement to the next should be very small, and each measurement should not deviate far from the mean. If one is less careful, each measurement will likely deviate much more, and then the difference of any measurement from the mean will generally be larger. The standard deviation in the first (careful) case will be small, and the standard deviation in the second case (not careful) will be large, reflecting the amount of scatter "about the mean" in the two sets of data. Technically, the standard deviation is the uncertainty due to randomness which is associated with each individual measurement made.

3.5 Significant Figures

Lastly, a word on significant figures. When one reads a number from an instrument, the number has a certain number of digits. For example, our stop watches will read a number like $00:01.33\,s$, or $00:12.45\,s$. In the first case, there are three significant digits, and in the second case there are four. The leading zeroes are meaningless and are not significant. Notice, however, that one could record a time of $00:02.20\,s$. This should be written down in your data table as $2.20\,s$, with the last zero included because it is a real digit readable off the stop watch and distinguishes this time from $2.19\,s$ and $2.21\,s$.

4 Procedure

Using the Saxon Bowls and an electronic stopwatch, you will time four droppings of each bowl. Start with the bowl's hole down and just making contact with the water. Drop the bowl and simultaneously start the timer. Stop the timer when the top edge of the bowl dips beneath the surface of the water. Do a total of six bowls with nicely separated hole sizes.

5 Calculations

Calculate the mean time for each bowl to drop, and then calculate the std. deviation, std. error, and finally the associated error on the mean for each. For this lab we want to see all of the calculations in long form. Put the *results* all in a neat table, with units clearly indicated at the top of each column.

Make a graph of the time to fall versus hole diameter. What does this look like? If it is not linear, try to find the mathematical relationship between time to fall and hole diameter using a graphing technique from the first lab.

Make and plot this linear graph, do a linear regression, and compare the fit parameters with any theoretical values you can determine.

6 Questions

1. Two data sets, each with the same number of trials for the same time measurement of a system, produce standard deviations of $0.23\,s$ and $0.76\,s$. What can you say about the data sets that produced each of these?
2. Is a mean value from a data set with a small error necessarily correct? Why/why not?
3. Suppose that the holes in the bottom of the bowls were square shaped instead of circular. How would this have changed your results? What would your first graph have looked like? What would a ln-ln graph have for a slope?

7 Error Analysis Data Table

Cap #	Hole Dia. (cm)	Time 1 (s)	Time 2 (s)	Time 3 (s)	Time 4	Avg (s)